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Algorithm Theory - Winter Term 2017/2018
Exercise Sheet 5

Hand in by Thursday 10:15, January 11, 2017

Exercise 1: Maximal Matching (4+6 Points)

Consider the following simple algorithm to find a maximal matching in a given graph $G = (V, E)$. Consider an initially empty set M. Pick an arbitrary edge $\{u, v\} \in E$ and add it to M. Then, remove all the edges adjacent to u or v from E. Repeat adding edges from E to M, as explained, until E becomes empty.

(a) Show that the algorithm computes a matching of size at least half the size of an optimal matching.

Now, let us assume that each edge e in the given graph $G = (V, E)$ is assigned a positive integer w_e as its weight.

(b) Provide a greedy algorithm (by adapting the above algorithm) to find a maximal matching with weight at least half of the weight of an optimal matching. Show why the solution is within factor 2 of an optimal solution.

Sample Solution

(a) Consider an arbitrary matching M in any given graph. Let $v(M)$ denote the set of the endpoints of the edges in M. Then, since each node in $v(M)$ is an endpoint of one and only one edge in M, $|v(M)| = 2 \cdot |M|$. Let us assume that M^* is an optimal matching in G. Therefore, we have $|v(M)| = 2 \cdot |M|$ and $|v(M^*)| = 2 \cdot |M^*|$.

Consider an arbitrary edge $e = \{u, w\} \in M^*$. If $e \in M$, then both u and v are in $v(M)$. Otherwise, at least one of u or w is in $v(M)$. This is due to the fact that M is maximal and if none of the endpoints of e were in $v(M)$, then e would have been added by the algorithm. Therefore, it holds that $v(M^*) \leq 2 \cdot v(M)$, which proves $|M^*| \leq 2 \cdot |M|$.

(b) The algorithm is as follows. Start with an empty set M. Pick an edge $e \in E$ with maximum weight. Add e to M and remove all adjacent edges of the two endpoints of e from E (which includes e itself). Repeat this step until E becomes empty.

Let $s = |M|$. The algorithm repeats the edge removal step s times. Let $e_i \in M$ be the i^{th} edge that the algorithm adds to M and let R_i be the set of adjacent edges to e_i that are removed in the i^{th} step of the algorithm (including e_i). The sets $R_1, R_2, \ldots, R_s$ form a partition of E. Let M^* be the maximum weighted Matching. We compare $M \cap R_i$ and $M^* \cap R_i$.

We know that $w(e_i)$ is bigger equal than the weight of any other edge from R_i adjacent to e_i (otherwise our greedy algorithm would have picked it instead of e_i). Moreover, $|M^* \cap R_i|$ is at most two times $|M \cap R_i| = 1$ (same argument as in (a)). Take these two arguments together and we have that $w(M^* \cap R_i) \leq 2w(M \cap R_i)$ (for $E' \subseteq E$ define $w(E') := \sum_{e \in E'} w(e)$).

Finally we have $w(M^*) = \sum_{i=1}^{s} w(M^* \cap R_i) \leq \sum_{i=1}^{s} 2w(M \cap R_i) = 2w(M)$.

Exercise 2: Perfect Matching (5 Points)

For a positive integer r, an r-regular graph is a graph where each node has the same degree r. Show that any r-regular bipartite graph has a perfect matching.

Sample Solution

Let G be a r-regular graph with bipartition $V(G) = U \cup V$. Note that $|U| = |V|$, since

$$r \cdot |U| = \sum_{u \in U} deg(u) = |E| = \sum_{v \in V} deg(v) = r \cdot |V|.$$

Consider an arbitrary set $X \subseteq U$. Let $N(X)$ denote the set of the neighbors of X in V. The number of edges from X to $N(X)$ is $r \cdot |X|$. The number of edges from $N(X)$ to U is $r \cdot |N(X)|$.
The edges from X to $N(X)$ are a subset of the edges $N(X)$ to U, hence the number of edges from X to $N(X)$ is less or equal to the number of edges from $N(X)$ to U.
So we have $r \cdot |X| \leq r \cdot |N(X)|$, which implies $|X| \leq |N(X)|$. Due to the Hall's Theorem, we can conclude that G has a perfect matching.

Exercise 3: Ford Fulkerson Revisited (10 Points)

Show that the below statement is correct or prove that it does not hold.
Often the Ford Fulkerson algorithm needs to consider many augmenting paths. If the algorithm always chooses the 'correct' augmenting paths it never has to choose more than $|E|$ paths.

Sample Solution

Let $G = (V, E)$ be a flow network with max flow $f : E \to \mathbb{R}^+$. In the following we show the existence of at most $|E|$ augmenting paths which form the max flow f. To construct these paths we make use of the max flow f. Note that this approach is actually not helpful for an algorithm because it has to know the max flow f in advance in order to determine the at most $|E|$ augmenting paths.

Construction of One Augmenting Path: Let $G(f) = (V, E_f)$ where $E_f = \{d \in E \mid f(d) > 0\}$. If $|f| = 0$ the graph $G(f)$ does not have any edges and the claim holds. If $|f| > 0$ then there is a path from s to t in $G(f)$. Pick any such path and denote it by P. Then let e be an edge on the path with smallest flow $f(e) = \min\{f(d) \mid d \text{ is edge on } P\}$. Now construct the augmenting path P such that it has the maximum flow $f(e)$.

Iterating the Construction: Redefine the flow network by reducing all capacities of G on the path P by $f(e)$. This way one obtains a new flow network with max flow $|f| - f(e)$ which is met by a flow f' which we define as the flow f reduced by the first augmenting path. To obtain the second augmenting path we again look at the induced graph $G(f')$ and proceed as before. The crucial observation is that $G(f')$ lost edge e (and we are done if $G(f')$ does not have any edge left). Thus we repeat this procedure at most $|E|$ times and in the end all $|E|$ augmenting paths combined form the max flow of the original flow network.

Exercise 4: Large Chromatic Number without Cliques (1+5+5+3+1 Points)

A c-coloring of a graph $G = (V, E)$ is a function $\phi : V \to \{1, \ldots, c\}$ such that any two neighboring nodes have different colors, i.e., for each $\{u, v\} \in E$, $\phi(u) \neq \phi(v)$. The chromatic number $\chi(G)$ of a graph G is the smallest integer c such that a c-coloring of G exists, e.g., the chromatic number of a k-node clique is k. In the following we use probability theory to show that not only cliques imply large chromatic number, in particular we would like to show the following:

For any k and l there is a graph with chromatic number greater than k and no cycle shorter than l.

In the following consider a (random) graph $G_{n,p}$ on n nodes, where each (possible) edge $\{u, v\}, u, v \in V$ exists with probability $p = n^{\frac{1}{2l}-1}$.

(a) An independent set I of a graph G is a set of nodes such that no two nodes in I are neighbors in G. The independence number $\alpha(G)$ of a graph denotes the size of the largest independent set.

Explain why $\chi(G) \geq |V(G)|/\alpha(G)$ holds.

(b) Show that for $a = \lceil \frac{3}{p} \ln n \rceil$ we have

$$\Pr[\alpha(G) \geq a] \longrightarrow_{n\to\infty} 0.$$

Hint: There are $\binom{n}{a}$ choices for subset of V with size a. What is the probability that a specific set of nodes of size a form an independent set? Also use the linearity of expectation!

(c) Let X be the number of cycles of length at most l. For large n, show that $E[X]$ can be upper bounded by $\frac{n}{4}$.

Hint: What is the probability that j specific nodes form a cycle? How many choices of nodes that can possibly form a cycle of length less than l are there? Again, use the linearity of expectation.

(d) From (*b*) and (*c*), we can deduce that $\Pr[X \geq n/2 \text{ or } \alpha(G) \geq a] < 1$ holds. This means that there exists a graph H with n nodes where the number of cycles with length less than l is less than $n/2$ and the independence number is smaller than a. So H has a small independence number but it might contain some short cycles.

Explain how to modify the graph H to obtain a graph H' with no cycles of length at most l, $\alpha(H') < a$ and $|V(H')| \geq n/2$.

(e) Show that the graph H' has no cycle of length at most l and a chromatic number at least k.

Remark: All subquestions in this exercise can be solved independently from each other (by using the results of the other questions as black box).

Sample Solution

We first fix the parameters k and l and then do the following steps to find a graph which has chromatic number larger than k and does not have cycles shorter than l. Note that k and l cannot be a function of the number of nodes as n is chosen sufficiently large in many of the following steps where the *sufficiently large* depends on k and l.

(a) Every color class of a valid coloring forms an independent set. Thus no color class can contain more than $\alpha(G)$ nodes which implies that there have to be at least $\frac{|V(G)|}{\alpha(G)}$ color classes.

(b) The probability that a given set of a nodes forms an independent set is $(1-p)^{\binom{a}{2}}$. There are $\binom{n}{a}$ to pick sets of a nodes from n nodes. With an union bound we obtain

$$\begin{aligned}
\Pr[\alpha(G) \geq a] &= \Pr[\exists W \subseteq V, W \text{ independent set}, |W| \geq a] \\
&= \Pr[\exists W \subseteq V, W \text{ independent set}, |W| = a] \\
&\leq \sum_{W \subseteq V, |W|=a} \Pr[W \text{ is an independent set}] \\
&\leq \binom{n}{a}(1-p)^{\binom{a}{2}} \\
&\leq n^a e^{-pa(a-1)/2} && \big(1+x \leq e^x \text{ for } x \in \mathbb{R}\big) \\
&\leq \frac{n^a}{n^{\frac{3}{2}(a-1)}} \overset{n\to\infty}{\longrightarrow} 0. && \big(\text{we can assume } a \geq 3 \text{ since } a \text{ grows arbitrarily large in } n\big)
\end{aligned}$$

(c) Let X_j be the number of cycles in G of length exactly j. Then we have $X = \sum_{j=3}^{\ell} X_j$. If we choose a series of nodes $(v_1, \dots, v_j)$ then the probability these nodes form a cycle in exactly that order is p^j. The number of series of nodes of length j is at most n^j. Hence we have $\mathbb{E}[X_j] \leq n^j p^j$.

$$\mathbb{E}[X] = \sum_{j=3}^{\ell} \mathbb{E}[X_j] \leq \sum_{j=3}^{\ell} n^j p^j = \sum_{j=3}^{\ell} n^{\frac{1}{2\ell} j} \leq \sum_{j=0}^{\ell} n^{\frac{1}{2\ell} j} \qquad \text{(geometric series)}$$

$$= \frac{1 - n^{\frac{\ell+1}{2\ell}}}{1 - n^{\frac{1}{2\ell}}} = \frac{n^{-\frac{1}{2\ell}} - n^{\frac{\ell}{2\ell}}}{n^{-\frac{1}{2\ell}} - 1} = \frac{n^{\frac{1}{2}} - n^{-\frac{1}{2\ell}}}{1 - n^{-\frac{1}{2\ell}}} \leq \frac{n^{\frac{1}{2}}}{1 - n^{-\frac{1}{2\ell}}} = \frac{n}{n^{\frac{1}{2}}(1 - n^{-\frac{1}{2\ell}})}$$

For large enough n, this is smaller than $\frac{n}{4}$ (we get a dependency of n on ℓ).

(d) The graph H has at most $n/2$ cycles of length at most ℓ and independence number $\alpha(H) < a$. We obtain H' by removing one node from each of these cycles. Removing a node from a graph can not increase the independence number. Then the graph H' has at least $n/2$ nodes, no cycles shorter than ℓ and independence number $\alpha(H') < a$.

(e) The graph H' has the following chromatic number.

$$\chi(H') \geq \frac{|V(H')|}{\alpha(H')} \geq \frac{n/2}{a} \geq \frac{n/2}{3n^{1-\frac{1}{2\ell}} \ln n} = \frac{\sqrt[2\ell]{n}}{6 \ln n}. \qquad \left(a = \lceil \tfrac{3}{p} \ln n \rceil, p = n^{\frac{1}{2\ell} - 1}\right)$$

If we chose n sufficiently large we obtain $\chi(H') > k$ (here we get that n depends on k).

Remark: The above proof was a probabilistic proof which shows that such graphs exist. However, it is very hard to actually construct any of these graphs.